

GEOMETRIC OBJECTS ASSOCIATED WITH THE FUNDAMENTAL CONNECTIONS IN FINSLER GEOMETRY

Nabil L. Youssef[†], S. H. Abed[†] and A. Soleiman[‡]

[†]Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt
nlyoussef2003@yahoo.fr, sabet52@yahoo.fr

and

[‡]Department of Mathematics, Faculty of Science,
Benha University, Benha, Egypt
soleiman@mail.eun.eg

Abstract. The aim of the present paper is to provide an *intrinsic* investigation of the properties of the most important geometric objects associated with the fundamental linear connections in Finsler geometry. We investigate intrinsically the most general relations concerning the torsion tensor fields and the curvature tensor fields associated with a given regular connection on the pullback bundle of a Finsler manifold. These relations, in turn, play a key role in obtaining other interesting results concerning the properties of the most important geometric objects associated with the fundamental canonical linear connections on the pullback bundle of a Finsler manifold, namely, the Cartan connection, the Berwald connection, the Chern (Rund) connection and the Hashiguchi connection.

For the sake of completeness and for comparison reasons, we provide an appendix presenting a global survey of canonical linear connections in Finsler geometry and the fundamental geometric objects associated with them.

Keywords: Regular connection, Barthel connection, Cartan connection, Berwald connection, Chern connection, Hashiguchi connection, Torsion tensor field, Curvature tensor field.

2000 AMS Subject Classification. 53C60, 53B40

Introduction

Studying Finsler geometry, one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

In Riemannian geometry, there is a canonical linear connection on the manifold M , namely, the Levi-Civita connection, whereas in Finsler geometry there is a corresponding canonical linear connection due to E. Cartan. However, this is not a connection on M but is a connection on $T(TM)$, the tangent bundle of TM , or on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : TM \rightarrow M$. Moreover, in Riemannian geometry there is one curvature tensor and no torsion tensor associated with the Levi-Civita connection on M , whereas in Finsler geometry there are three curvature tensors and five torsion tensors associated with the Cartan connection on $\pi^{-1}(TM)$. Besides, there are other canonical linear connections together with their associated torsion and curvature tensor fields. Consequently, Finsler geometry is richer in structure and content than Riemannian geometry and thus potentially more appropriate for dealing with physical theories at a deeper level.

The theory of connections and their associated geometric objects is an important field of differential geometry. The most important linear connections and their associated geometric objects in Finsler geometry were studied *locally* in [2], [8], [9],...etc.

In [12] and [13], we have provided new intrinsic proofs of intrinsic versions of the existence and uniqueness theorems for the fundamental linear connections on the pullback bundle of a Finsler manifold, namely, the Cartan connection, the Berwald connection, the Chern (Rund) connection and the Hashiguchi connection. The present paper is a continuation of this work where we investigate *intrinsically* the fundamental properties of the most important geometric objects associated with these connections.

The paper consists of five parts preceded by an introductory section (§1), which provides a brief account of the basic concepts and results necessary for this work. For more details, we refer to [1], [3], [4], [7], and [10].

In the first part (§2), we investigate the fundamental relations concerning the torsion tensor fields and the curvature tensor fields associated with a given regular connection on $\pi^{-1}(TM)$. These relations, in turn, play a key role in obtaining other interesting results. The second part (§3) is devoted to study the fundamental properties of the most important geometric objects associated with the Cartan connection. In the third part (§4), various fundamental relations and properties concerning the torsion tensor fields and the curvature tensor fields associated with the Berwald connection are obtained. In the fourth and the fifth parts ((§5) and (§6)), as in the previous sections, we study the most important geometric objects associated with the Chern connection and the Hashiguchi connection, respectively.

For the sake of completeness and for comparison reasons, the paper is concluded with an appendix presenting a global survey of canonical linear connections in Finsler geometry and the fundamental geometric objects associated with them.

The present work is formulated in a prospective modern coordinate-free form, without being trapped into the complications of indices. However, the local expressions of the obtained results, when calculated, coincide with the existing classical local results.

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1],[3],[7] and [10]. We make the assumption that the geometric objects we consider are of class C^∞ .

The following notation will be used throughout this paper:

M : a real paracompact differentiable manifold of finite dimension n and of class C^∞ ,

$\mathfrak{F}(M)$: the \mathbb{R} -algebra of differentiable functions on M ,

$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$ -module of vector fields on M ,

$\pi_M : TM \longrightarrow M$: the tangent bundle of M ,

$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M ,

$V(TM)$: the vertical subbundle of the bundle TTM ,

$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π ,

$\mathfrak{X}(\pi^{-1}(TM))$: the $\mathfrak{F}(\mathcal{T}M)$ -module of differentiable sections of $\pi^{-1}(TM)$,

i_X : the interior product with respect to $X \in \mathfrak{X}(M)$,

df : the exterior derivative of $f \in \mathfrak{X}(M)$,

$d_L := [i_L, d]$, i_L being the interior derivative with respect to a vector form L .

Elements of $\mathfrak{X}(\pi^{-1}(TM))$ will be called π -vector fields and will be denoted by barred letters \bar{X} . Tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM . The vertical vector field \mathcal{C} on TM defined by $\mathcal{C} := \gamma \circ \bar{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map

$$K : T(\mathcal{T}M) \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X \bar{\eta},$$

called the connection (or the deflection) map of D . A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \quad \forall u \in \mathcal{T}M. \quad (1.1)$$

If M is endowed with a regular connection, then the vector bundle maps

$$\begin{aligned} \gamma &: \pi^{-1}(TM) \longrightarrow V(\mathcal{T}M), \\ \rho|_{H(\mathcal{T}M)} &: H(\mathcal{T}M) \longrightarrow \pi^{-1}(TM), \\ K|_{V(\mathcal{T}M)} &: V(\mathcal{T}M) \longrightarrow \pi^{-1}(TM) \end{aligned}$$

are vector bundle isomorphisms. Let us denote $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, then

$$\rho \circ \beta = id_{\pi^{-1}(TM)}, \quad \beta \circ \rho = \begin{cases} id_{H(TM)} & \text{on } H(TM) \\ 0 & \text{on } V(TM) \end{cases} \quad (1.2)$$

The map β will be called the horizontal map of the connection D .

According to the direct sum decomposition (1.1), a regular connection D gives rise to a horizontal projector h_D and a vertical projector v_D , given by

$$h_D = \beta \circ \rho, \quad v_D = I - \beta \circ \rho, \quad (1.3)$$

where I is the identity endomorphism on $T(TM)$: $I = id_{T(TM)}$.

The (classical) torsion tensor \mathbf{T} of the connection D is defined by

$$\mathbf{T}(X, Y) = D_X \rho Y - D_Y \rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(TM).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted by Q and T respectively, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta \overline{X} \beta \overline{Y}), \quad T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma \overline{X}, \beta \overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$T(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z}). \quad (1.4)$$

The (classical) curvature tensor \mathbf{K} of the connection D is defined by

$$\mathbf{K}(X, Y) \rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X, Y]} \rho Z \quad \forall X, Y, Z \in \mathfrak{X}(TM).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R , P and S respectively, are defined by

$$R(\overline{X}, \overline{Y}) \overline{Z} = \mathbf{K}(\beta \overline{X} \beta \overline{Y}) \overline{Z}, \quad P(\overline{X}, \overline{Y}) \overline{Z} = \mathbf{K}(\beta \overline{X}, \gamma \overline{Y}) \overline{Z}, \quad S(\overline{X}, \overline{Y}) \overline{Z} = \mathbf{K}(\gamma \overline{X}, \gamma \overline{Y}) \overline{Z}.$$

The contracted curvature tensors, denoted by \widehat{R} , \widehat{P} and \widehat{S} respectively, are also known as the (v)h-, (v)hv- and (v)v-torsion tensors and are defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y}) \overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y}) \overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y}) \overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y}) \overline{Z}, \overline{W}), \dots, S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(S(\overline{X}, \overline{Y}) \overline{Z}, \overline{W}). \quad (1.5)$$

$$\widehat{R}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}), \dots, \widehat{S}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{S}(\overline{X}, \overline{Y}), \overline{Z}). \quad (1.6)$$

We terminate this section by some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [4], [5] and [6].

A semispray is a vector field X on TM , C^∞ on $\mathcal{T}M$, C^1 on TM , such that $\rho \circ X = \overline{\eta}$. A semispray X which is homogeneous of degree 2 in the directional argument ($[C, X] = X$) is called a spray.

Proposition 1.1. [6] *Let (M, L) be a Finsler manifold. The vector field G on TM defined by $i_G \Omega = -dE$ is a spray, where $E := \frac{1}{2} L^2$ is the energy function and $\Omega := dd_J E$. Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector 1-form Γ on TM , C^∞ on $\mathcal{T}M$, C^0 on TM , such that

$$J\Gamma = J, \quad \Gamma J = -J.$$

The horizontal and vertical projectors h_Γ and v_Γ associated with Γ are defined by $h_\Gamma := \frac{1}{2}(I + \Gamma)$ and $v_\Gamma := \frac{1}{2}(I - \Gamma)$.

Theorem 1.2. [5] *On a Finsler manifold (M, L) , there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by:*

$$\Gamma = [J, G],$$

where G is the canonical spray.

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M, L) .

2. Fundamental identities associated with regular connections

In this section, we investigate the most important general properties concerning the torsion and curvature tensor fields associated with regular connections on $\pi^{-1}(TM)$. These properties will play a key role throughout the whole paper.

Definition 2.1. [12] *Let D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β .*
– *The semispray $S = \beta \circ \bar{\eta}$ will be called the semispray associated with D .*
– *The nonlinear connection $\Gamma = 2\beta \circ \rho - I$ will be called the nonlinear connection associated with D and will be denoted by Γ_D .*

Proposition 2.2. [12] *Let D be a regular connection on $\pi^{-1}(TM)$ whose connection map is K and whose horizontal map is β . Then, the following assertions are equivalent:*

- (a) *The (h)hv-torsion T of D has the property that $T(\bar{X}, \bar{\eta}) = 0$,*
- (b) *$K = \gamma^{-1}$ on $V(TM)$,*
- (c) *$\Gamma := \beta \circ \rho - \gamma \circ K$ is a nonlinear connection on M .*

Consequently, if any one of the above assertions holds, then Γ coincides with the nonlinear connection associated with D : $\Gamma = \Gamma_D = 2\beta \circ \rho - I$, and in this case $h_\Gamma = h_D = \beta \circ \rho$ and $v_\Gamma = v_D = \gamma \circ K$.

The following two lemmas are fundamental for subsequent use.

Lemma 2.3. *Let D be a regular connection on $\pi^{-1}(TM)$ whose (h)hv-torsion tensor T has the property that $T(\bar{X}, \bar{\eta}) = 0$. Then, we have:*

- (a) $[\beta\bar{X}, \beta\bar{Y}] = \gamma\hat{R}(\bar{X}, \bar{Y}) + \beta(D_{\beta\bar{X}}\bar{Y} - D_{\beta\bar{Y}}\bar{X} - Q(\bar{X}, \bar{Y}))$,
- (b) $[\gamma\bar{X}, \beta\bar{Y}] = -\gamma(\hat{P}(\bar{Y}, \bar{X}) + D_{\beta\bar{Y}}\bar{X}) + \beta(D_{\gamma\bar{X}}\bar{Y} - T(\bar{X}, \bar{Y}))$,
- (c) $[\gamma\bar{X}, \gamma\bar{Y}] = \gamma(D_{\gamma\bar{X}}\bar{Y} - D_{\gamma\bar{Y}}\bar{X} + \hat{S}(\bar{X}, \bar{Y}))$.

Proof. It should first be noted that, as D is regular and $T(\overline{X}, \overline{\eta}) = 0$, we have $h = \beta \circ \rho$, $v = \gamma \circ K$, $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, by Proposition 2.2. Then, we have

$$\begin{aligned} [\beta\overline{X}, \beta\overline{Y}] &= \gamma(K[\beta\overline{X}, \beta\overline{Y}]) + \beta(\rho[\beta\overline{X}, \beta\overline{Y}]) = \gamma(D_{[\beta\overline{X}, \beta\overline{Y}]} \overline{\eta}) + \beta(\rho[\beta\overline{X}, \beta\overline{Y}]) \\ &= \gamma(\widehat{R}(\overline{X}, \overline{Y}) - D_{\beta\overline{Y}} D_{\beta\overline{X}} \overline{\eta} + D_{\beta\overline{X}} D_{\beta\overline{Y}} \overline{\eta}) + \beta(D_{\beta\overline{X}} \overline{Y} - D_{\beta\overline{Y}} \overline{X} - Q(\overline{X}, \overline{Y})) \\ &= \gamma \widehat{R}(\overline{X}, \overline{Y}) + \beta(D_{\beta\overline{X}} \overline{Y} - D_{\beta\overline{Y}} \overline{X} - Q(\overline{X}, \overline{Y})). \end{aligned}$$

On the other hand,

$$\begin{aligned} [\gamma\overline{X}, \beta\overline{Y}] &= \gamma(K[\gamma\overline{X}, \beta\overline{Y}]) + \beta(\rho[\gamma\overline{X}, \beta\overline{Y}]) = \gamma(D_{[\gamma\overline{X}, \beta\overline{Y}]} \overline{\eta}) + \beta(\rho[\gamma\overline{X}, \beta\overline{Y}]) \\ &= -\gamma(\widehat{P}(\overline{Y}, \overline{X}) + D_{\beta\overline{Y}} D_{\gamma\overline{X}} \overline{\eta} - D_{\gamma\overline{X}} D_{\beta\overline{Y}} \overline{\eta}) + \beta(D_{\gamma\overline{X}} \overline{Y} - T(\overline{X}, \overline{Y})) \\ &= -\gamma(\widehat{P}(\overline{Y}, \overline{X}) + D_{\beta\overline{Y}} \overline{X}) + \beta(D_{\gamma\overline{X}} \overline{Y} - T(\overline{X}, \overline{Y})). \end{aligned}$$

The last identity can be proved analogously. \square

Lemma 2.4. *Let D be a linear connection on $\pi^{-1}(TM)$ with (classical) torsion tensor \mathbf{T} and (classical) curvature tensor \mathbf{K} . For every $X, Y, Z \in \mathfrak{X}(TM)$, $\overline{Z}, \overline{W} \in \mathfrak{X}(\pi(M))$, we have:*

(a) $\mathfrak{S}_{X,Y,Z}\{\mathbf{K}(X, Y)\rho Z + D_X \mathbf{T}(Y, Z) + \mathbf{T}(X, [Y, Z])\} = 0$,

(b) $\mathfrak{S}_{X,Y,Z}\{D_Z \mathbf{K}(X, Y)\overline{W} - \mathbf{K}(X, Y)D_Z \overline{W} - \mathbf{K}([X, Y], Z)\overline{W}\} = 0$.

If $\pi^{-1}(TM)$ is equipped with a metric g , then

(c) $g(\mathbf{K}(X, Y)\overline{Z}, \overline{W}) + g(\mathbf{K}(X, Y)\overline{W}, \overline{Z}) = \mathfrak{U}_{X,Y}\{(D_X(D_Y g))(\overline{W}, \overline{Z})\} - (D_{[X,Y]} g)(\overline{W}, \overline{Z})$.

Proof. We prove (c) only.

$$X \cdot g(\overline{W}, \overline{Z}) = (D_X g)(\overline{W}, \overline{Z}) + g(D_X \overline{W}, \overline{Z}) + g(\overline{W}, D_X \overline{Z}).$$

From which, we obtain

$$\begin{aligned} X \cdot (Y \cdot g(\overline{W}, \overline{Z})) &= X \cdot ((D_Y g)(\overline{W}, \overline{Z})) + (D_X g)(D_Y \overline{W}, \overline{Z}) + (D_X g)(\overline{W}, D_Y \overline{Z}) \\ &\quad + g(D_X D_Y \overline{W}, \overline{Z}) + g(D_Y \overline{W}, D_X \overline{Z}) + g(D_X \overline{W}, D_Y \overline{Z}) \\ &\quad + g(\overline{W}, D_X D_Y \overline{Z}), \end{aligned}$$

with similar expression for $Y \cdot (X \cdot g(\overline{W}, \overline{Z}))$. Consequently,

$$\begin{aligned} [X, Y] \cdot g(\overline{W}, \overline{Z}) &= \mathfrak{U}_{X,Y}\{X \cdot ((D_Y g)(\overline{W}, \overline{Z})) + (D_X g)(D_Y \overline{W}, \overline{Z}) + (D_X g)(\overline{W}, D_Y \overline{Z})\} \\ &\quad + g([D_X, D_Y] \overline{W}, \overline{Z}) + g(\overline{W}, [D_X, D_Y] \overline{Z}). \end{aligned}$$

On the other hand, we have

$$[X, Y] \cdot g(\overline{W}, \overline{Z}) = (D_{[X,Y]} g)(\overline{W}, \overline{Z}) + g(D_{[X,Y]} \overline{W}, \overline{Z}) + g(\overline{W}, D_{[X,Y]} \overline{Z})$$

The result follows from the above two equations. \square

Proposition 2.5. *Let D be a regular connection on $\pi^{-1}(TM)$ whose (h)hv-torsion tensor T has the property that $T(\overline{X}, \overline{\eta}) = 0$. Then, we have:*

- (a) $S(\overline{X}, \overline{Y})\overline{Z} = (D_{\gamma\overline{Y}}T)(\overline{X}, \overline{Z}) - (D_{\gamma\overline{X}}T)(\overline{Y}, \overline{Z})$
 $+ T(\overline{X}, T(\overline{Y}, \overline{Z})) - T(\overline{Y}, T(\overline{X}, \overline{Z})) + T(\widehat{S}(\overline{X}, \overline{Y}), \overline{Z}),$
- (b) $P(\overline{X}, \overline{Y})\overline{Z} - P(\overline{Z}, \overline{Y})\overline{X} = (D_{\beta\overline{Z}}T)(\overline{Y}, \overline{X}) - (D_{\beta\overline{X}}T)(\overline{Y}, \overline{Z}) - (D_{\gamma\overline{Y}}Q)(\overline{X}, \overline{Z})$
 $- T(\overline{Y}, Q(\overline{X}, \overline{Z})) - T(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X}) + T(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$
 $- Q(\overline{Z}, T(\overline{Y}, \overline{X})) + Q(\overline{X}, T(\overline{Y}, \overline{Z})),$
- (c) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{R(\overline{X}, \overline{Y})\overline{Z} - T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\} = \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{Q(\overline{X}, Q(\overline{Y}, \overline{Z})) - (D_{\beta\overline{X}}Q)(\overline{Y}, \overline{Z})\}.$

Proof. Follows from Lemma 2.4(a) and Lemma 2.3. \square

Proposition 2.6. *Let D be a regular connection on $\pi^{-1}(TM)$ whose (h)hv-torsion tensor T has the property that $T(\overline{X}, \overline{\eta}) = 0$. Then, we have:*

- (a) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(D_{\gamma\overline{X}}S)(\overline{Y}, \overline{Z}, \overline{W}) - S(\widehat{S}(\overline{X}, \overline{Y}), \overline{Z})\overline{W}\} = 0.$
- (b) $(D_{\beta\overline{Z}}S)(\overline{X}, \overline{Y}, \overline{W}) - (D_{\gamma\overline{X}}P)(\overline{Z}, \overline{Y}, \overline{W}) + (D_{\gamma\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) =$
 $= P(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W} - P(T(\overline{Y}, \overline{Z}), \overline{X})\overline{W} - P(\overline{Z}, \widehat{S}(\overline{X}, \overline{Y}))\overline{W}$
 $+ S(\widehat{P}(\overline{Z}, \overline{X}), \overline{Y})\overline{W} - S(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X})\overline{W}.$
- (c) $(D_{\gamma\overline{X}}R)(\overline{Y}, \overline{Z}, \overline{W}) + (D_{\beta\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) - (D_{\beta\overline{Z}}P)(\overline{Y}, \overline{X}, \overline{W}) =$
 $= P(\overline{Z}, \widehat{P}(\overline{Y}, \overline{X}))\overline{W} - P(\overline{Y}, \widehat{P}(\overline{Z}, \overline{X}))\overline{W} - P(Q(\overline{Y}, \overline{Z}), \overline{X})\overline{W}$
 $+ R(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W} - R(T(\overline{X}, \overline{Y}), \overline{Z})\overline{W} + S(\widehat{R}(\overline{Y}, \overline{Z}), \overline{X})\overline{W}.$
- (d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(D_{\beta\overline{X}}R)(\overline{Y}, \overline{Z}, \overline{W}) + P(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W} + R(Q(\overline{X}, \overline{Y}), \overline{Z})\overline{W}\} = 0.$

Proof. Follows from Lemma 2.4(b) and Lemma 2.3. \square

3. Fundamental tensors associated with the Cartan connection

We shall use the results obtained in §2 to investigate the fundamental properties of the most important tensors associated with Cartan connection.

Theorem 3.1. [12] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

- (i) ∇ is metric: $\nabla g = 0,$
- (ii) The (h)h-torsion of ∇ vanishes: $Q = 0,$
- (iii) The (h)hv-torsion T of ∇ satisfies $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y}).$

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L) .

Theorem 3.2. [12] *The nonlinear connection associated with the Cartan connection ∇ coincides with the Barthel connection: $\Gamma_{\nabla} = [J, G].$*

Proposition 3.3. *The (h)hv-torsion T of the Cartan connection has the properties:*

- (a) $T(\overline{X}, \overline{Y}, \overline{Z}) = T(\overline{X}, \overline{Z}, \overline{Y})$,
- (b) $(\nabla_W T)(\overline{X}, \overline{Y}, \overline{Z}) = g((\nabla_W T)(\overline{X}, \overline{Y}), \overline{Z}) = g((\nabla_W T)(\overline{X}, \overline{Z}), \overline{Y})$,
- (c) $T(\overline{X}, \overline{\eta}) = 0$,
- (d) $(\nabla_{\gamma \overline{X}} T)(\overline{Y}, \overline{Z}) = (\nabla_{\gamma \overline{Y}} T)(\overline{X}, \overline{Z})$,
- (e) $(\nabla_{\gamma \overline{\eta}} T)(\overline{X}, \overline{Y}) = -T(\overline{X}, \overline{Y})$,
- (f) T is totally symmetric.

Proof.

(b) Follows from the following relations, making use of (a):

$$g((\nabla_W T)(\overline{X}, \overline{Y}), \overline{Z}) = g(\nabla_W T(\overline{X}, \overline{Y}), \overline{Z}) - g(T(\nabla_W \overline{X}, \overline{Y}), \overline{Z}) - g(T(\overline{X}, \nabla_W \overline{Y}), \overline{Z}).$$

$$g((\nabla_W T)(\overline{X}, \overline{Z}), \overline{Y}) = g(\nabla_W T(\overline{X}, \overline{Z}), \overline{Y}) - g(T(\nabla_W \overline{X}, \overline{Z}), \overline{Y}) - g(T(\overline{X}, \nabla_W \overline{Z}), \overline{Y}).$$

$$g(\nabla_W T(\overline{X}, \overline{Y}), \overline{Z}) = W \cdot g(T(\overline{X}, \overline{Y}), \overline{Z}) - g(T(\overline{X}, \overline{Y}), \nabla_W \overline{Z}).$$

(c) As ∇ is a metric linear connection on $\pi^{-1}(TM)$ with nonzero torsion \mathbf{T} , one can show that ∇ is completely determined by the relation

$$\begin{aligned} 2g(\nabla_X \rho Y, \rho Z) &= X \cdot g(\rho Y, \rho Z) + Y \cdot g(\rho Z, \rho X) - Z \cdot g(\rho X, \rho Y) \\ &\quad - g(\rho X, \mathbf{T}(Y, Z)) + g(\rho Y, \mathbf{T}(Z, X)) + g(\rho Z, \mathbf{T}(X, Y)) \\ &\quad - g(\rho X, \rho[Y, Z]) + g(\rho Y, \rho[Z, X]) + g(\rho Z, \rho[X, Y]). \end{aligned} \quad (3.1)$$

for all $X, Y, Z \in \mathfrak{X}(TM)$. The connection ∇ being regular, let h and v be the horizontal and vertical projectors associated with the decomposition (1.3): $h = \beta \circ \rho$, $v = I - \beta \circ \rho$.

Replacing X, Y, Z by $\gamma \overline{X}, hY, hZ$ in (3.1) and using axiom (iii) of Theorem 3.1, taking into account the fact that $\rho \circ \gamma = 0$ and $\rho \circ h = \rho$, we get

$$2g(\nabla_{\gamma \overline{X}} \rho Y, \rho Z) = \gamma \overline{X} \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, \gamma \overline{X}]) + g(\rho Z, \rho[\gamma \overline{X}, hY]). \quad (3.2)$$

Now,

$$2g(T(\overline{X}, \overline{\eta}), \overline{Z}) = 2g(\mathbf{T}(\gamma \overline{X}, \beta \overline{\eta}), \overline{Z}) = 2g(\nabla_{\gamma \overline{X}} \overline{\eta}, \overline{Z}) - 2g(\rho[\gamma \overline{X}, \beta \overline{\eta}], \overline{Z}).$$

Then, from (3.2), we get

$$2g(T(\overline{X}, \overline{\eta}), \overline{Z}) = \gamma \overline{X} \cdot g(\overline{\eta}, \overline{Z}) + g(\overline{\eta}, \rho[\beta \overline{Z}, \gamma \overline{X}]) - g(\overline{Z}, \rho[\gamma \overline{X}, \beta \overline{\eta}]).$$

From which, together with the identity $\overline{X} = \rho[\gamma \overline{X}, \beta \overline{\eta}]$ [12], we obtain

$$2g(T(\overline{X}, \overline{\eta}), \overline{Z}) = \gamma \overline{X} \cdot g(\overline{\eta}, \overline{Z}) + g(\overline{\eta}, \rho[\beta \overline{Z}, \gamma \overline{X}]) - g(\overline{Z}, \overline{X}).$$

Finally, one can show that the sum of the first two terms on the right-hand side is equal to $g(\overline{X}, \overline{Z})$, from which the result.

(d) Since ∇ is regular with $T(\overline{X}, \overline{\eta}) = 0$, then, by Proposition 2.5(a) and property (a), we have

$$\begin{aligned} S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= g((\nabla_{\gamma \overline{Y}} T)(\overline{X}, \overline{Z}), \overline{W}) - g((\nabla_{\gamma \overline{X}} T)(\overline{Y}, \overline{Z}), \overline{W}) \\ &\quad + g(T(\overline{X}, \overline{W}), T(\overline{Y}, \overline{Z})) - g(T(\overline{Y}, \overline{W}), T(\overline{X}, \overline{Z})) \\ &\quad + g(T(\widehat{S}(\overline{X}, \overline{Y}), \overline{Z}), \overline{W}). \end{aligned}$$

On the other hand, using Lemma 2.4(c), together with axiom (i) of Theorem 3.1, we get

$$S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -S(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}). \quad (3.3)$$

Using the properties (a) and (b), the above two equations, yield

$$(\nabla_{\gamma\overline{X}}T)(\overline{Y}, \overline{Z}) - (\nabla_{\gamma\overline{Y}}T)(\overline{X}, \overline{Z}) = T(\widehat{S}(\overline{X}, \overline{Y}), \overline{Z}). \quad (3.4)$$

Substituting (3.4) in (a) of Proposition 2.5, we get

$$S(\overline{X}, \overline{Y})\overline{Z} = T(\overline{X}, T(\overline{Y}, \overline{Z})) - T(\overline{Y}, T(\overline{X}, \overline{Z})). \quad (3.5)$$

Setting $\overline{Z} = \overline{\eta}$ in (3.5) and noting that $T(\overline{X}, \overline{\eta}) = 0$, we have

$$\widehat{S}(\overline{X}, \overline{Y}) = 0. \quad (3.6)$$

Then, the result follows from (3.4) and (3.6) .

(f) Follows from (d) by setting $\overline{Z} = \overline{\eta}$, taking into account (c) and (a). \square

Theorem 3.4. *The v-curvature S of the Cartan connection has the properties :*

- (a) $S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -S(\overline{Y}, \overline{X}, \overline{Z}, \overline{W})$,
- (b) $S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -S(\overline{X}, \overline{Y}, \overline{W}, \overline{Z})$,
- (c) $S(\overline{X}, \overline{Y})\overline{Z} = T(\overline{X}, T(\overline{Y}, \overline{Z})) - T(\overline{Y}, T(\overline{X}, \overline{Z}))$,
- (d) $S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = g(T(\overline{X}, \overline{W}), T(\overline{Y}, \overline{Z})) - g(T(\overline{Y}, \overline{W}), T(\overline{X}, \overline{Z}))$,
- (e) $S(\overline{Z}, \overline{W}, \overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})$,
- (f) $S(\overline{X}, \overline{\eta})\overline{Y} = S(\overline{\eta}, \overline{X})\overline{Y} = \widehat{S}(\overline{X}, \overline{Y}) = 0$,
- (g) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(\nabla_{\gamma\overline{X}}S)(\overline{Y}, \overline{Z}, \overline{W})\} = 0$,
- (h) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{S(\overline{X}, \overline{Y})\overline{Z}\} = 0$,
- (i) $(\nabla_{\gamma\overline{\eta}}S)(\overline{X}, \overline{Y}, \overline{Z}) = -2S(\overline{X}, \overline{Y})\overline{Z}$,
- (j) $(\nabla_{\beta\overline{Z}}S)(\overline{X}, \overline{Y}, \overline{W}) = (\nabla_{\gamma\overline{X}}P)(\overline{Z}, \overline{Y}, \overline{W}) - (\nabla_{\gamma\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) - S(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X})\overline{W}$
 $+ S(\widehat{P}(\overline{Z}, \overline{X}), \overline{Y})\overline{W} - P(T(\overline{Y}, \overline{Z}), \overline{X})\overline{W} + P(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W}$.

Proof.

(b), (c) and (d) follow immediately from (3.3) and (3.5).

(e) and (f) follow from (d) and the properties of T .

(g) Follows from Proposition 2.6(a) and (3.6).

(h) and (i) follow from (g) by setting $\overline{W} = \overline{\eta}$ and $\overline{X} = \overline{\eta}$ respectively, taking (f) into account.

(j) Follows from Proposition 2.6(b) and (3.6). \square

Theorem 3.5. *The hv-curvature tensor P of the Cartan connection has the properties :*

- (a) $P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -P(\overline{X}, \overline{Y}, \overline{W}, \overline{Z})$,

- (b) $P(\overline{X}, \overline{Y})\overline{Z} - P(\overline{Z}, \overline{Y})\overline{X} = (\nabla_{\beta\overline{Z}}T)(\overline{Y}, \overline{X}) - (\nabla_{\beta\overline{X}}T)(\overline{Y}, \overline{Z}) - T(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X}) + T(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z}),$
- (c) $P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = g((\nabla_{\beta\overline{Z}}T)(\overline{X}, \overline{Y}), \overline{W}) - g((\nabla_{\beta\overline{W}}T)(\overline{X}, \overline{Y}), \overline{Z}) + g(T(\overline{X}, \overline{Z}), \widehat{P}(\overline{W}, \overline{Y})) - g(T(\overline{X}, \overline{W}), \widehat{P}(\overline{Z}, \overline{Y})),$
- (d) $\widehat{P}(\overline{\eta}, \overline{X}) = 0,$
- (e) $\widehat{P}(\overline{X}, \overline{Y}) = (\nabla_{\beta\overline{\eta}}T)(\overline{X}, \overline{Y}),$
- (f) \widehat{P} is symmetric,
- (g) $P(\overline{\eta}, \overline{X})\overline{Y} = P(\overline{X}, \overline{\eta})\overline{Y} = 0,$
- (h) $(\nabla_{\gamma\overline{\eta}}P)(\overline{X}, \overline{Y}, \overline{Z}) = -P(\overline{X}, \overline{Y})\overline{Z},$
- (i) $P(\overline{X}, \overline{Y})\overline{Z} = P(\overline{Y}, \overline{X})\overline{Z} - (\nabla_{\beta\overline{\eta}}S)(\overline{X}, \overline{Y}, \overline{Z}).$

Proof.

- (a) Follows from Lemma 2.4(c) by setting $X = \beta\overline{X}, Y = \gamma\overline{Y}$, noting that $\nabla g = 0$.
- (b) Follows from Proposition 2.5(b) and Theorem 3.1(ii).
- (c) From (b), making use of Proposition 3.3(a), we have

$$P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) - P(\overline{Z}, \overline{Y}, \overline{X}, \overline{W}) = g((\nabla_{\beta\overline{Z}}T)(\overline{Y}, \overline{X}), \overline{W}) - g((\nabla_{\beta\overline{X}}T)(\overline{Y}, \overline{Z}), \overline{W}) - g(T(\overline{X}, \overline{W}), \widehat{P}(\overline{Z}, \overline{Y})) + g(T(\overline{Z}, \overline{W}), \widehat{P}(\overline{X}, \overline{Y})).$$

By cyclic permutation on $\overline{X}, \overline{Z}, \overline{W}$ of the above equation, one gets

$$\begin{aligned} P(\overline{W}, \overline{Y}, \overline{X}, \overline{Z}) - P(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) &= g((\nabla_{\beta\overline{X}}T)(\overline{Y}, \overline{W}), \overline{Z}) - g((\nabla_{\beta\overline{W}}T)(\overline{Y}, \overline{X}), \overline{Z}) \\ &\quad - g(T(\overline{W}, \overline{Z}), \widehat{P}(\overline{X}, \overline{Y})) + g(T(\overline{X}, \overline{Z}), \widehat{P}(\overline{W}, \overline{Y})), \\ P(\overline{Z}, \overline{Y}, \overline{W}, \overline{X}) - P(\overline{W}, \overline{Y}, \overline{Z}, \overline{X}) &= g((\nabla_{\beta\overline{W}}T)(\overline{Y}, \overline{Z}), \overline{X}) - g((\nabla_{\beta\overline{Z}}T)(\overline{Y}, \overline{W}), \overline{X}) \\ &\quad - g(T(\overline{Z}, \overline{X}), \widehat{P}(\overline{W}, \overline{Y})) + g(T(\overline{W}, \overline{X}), \widehat{P}(\overline{Z}, \overline{Y})). \end{aligned}$$

Adding the first two equations and subtracting the third, using (a) and Proposition 3.3(b), (f), the result follows.

- (d) Follows from (c) by setting $\overline{X} = \overline{Z} = \overline{\eta}$, making use of the properties of T and the fact that $K \circ \beta = 0$.
- (e) Follows from (c) by setting $\overline{Z} = \overline{\eta}$, taking (d) and the properties of T into account.
- (f) Follows from (e) together with the symmetry of T .
- (g) Follows from (c) by setting $\overline{X} = \overline{\eta}$ (resp. $\overline{Y} = \overline{\eta}$), making use of the obtained properties of the (v)hv-torsion \widehat{P} and the (h)hv-torsion T .
- (h) Follows from the property (i) of the v-curvature tensor S (Theorem 3.4) by setting $\overline{X} = \overline{\eta}$ and making use of the obtained properties of T, S and P .
- (i) Can be proved in an analogous manner as (h). □

Theorem 3.6. *The h-curvature tensor R of the Cartan connection has the properties :*

- (a) $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R(\overline{Y}, \overline{X}, \overline{Z}, \overline{W}),$

- (b) $R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}),$
- (c) $\widehat{R}(\overline{X}, \overline{Y}) = -K\mathfrak{R}(\beta\overline{X}, \beta\overline{Y}),$ where \mathfrak{R} is the curvature of Barthel connection,
- (d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{R(\overline{X}, \overline{Y})\overline{Z} - T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\} = 0,$
- (e) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{(\nabla_{\beta\overline{X}}R)(\overline{Y}, \overline{Z}, \overline{W}) + P(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W}\} = 0,$
- (f) $(\nabla_{\gamma\overline{X}}R)(\overline{Y}, \overline{Z}, \overline{W}) + (\nabla_{\beta\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) - (\nabla_{\beta\overline{Z}}P)(\overline{Y}, \overline{X}, \overline{W})$
 $- P(\overline{Z}, \widehat{P}(\overline{Y}, \overline{X}))\overline{W} + R(T(\overline{X}, \overline{Y}), \overline{Z})\overline{W} - S(\widehat{R}(\overline{Y}, \overline{Z}), \overline{X})\overline{W}$
 $+ P(\overline{Y}, \widehat{P}(\overline{Z}, \overline{X}))\overline{W} - R(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W} = 0,$
- (g) $(\nabla_{\gamma\overline{\eta}}R)(\overline{X}, \overline{Y}, \overline{Z}) = 0.$

Proof.

(b) Follows from Lemma 2.4(c) by setting $X = \beta\overline{X}$ and $Y = \beta\overline{Y}$, taking into account the fact that $\nabla g = 0$.

(c) We use the identity $\mathfrak{R}(X, Y) = -v[hX, hY]$ [11] together with Lemma 2.3(a) and the fact that $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$ and $K \circ \beta = 0$:

$$\begin{aligned} v[hX, hY] &= \gamma \circ K[(\beta \circ \rho)X, (\beta \circ \rho)Y] \\ &= \gamma \circ K\{\gamma\widehat{R}(\rho X, \rho Y) + \beta(D_{hX}\rho Y - D_{hY}\rho X - Q(\rho X, \rho Y))\} \\ &= \gamma\widehat{R}(\rho X, \rho Y), \end{aligned}$$

from which the result.

(d) Follows from Proposition 2.5(c) and Theorem 3.1(b).

(e) and (f) follow from Proposition 2.6(d) and (c) respectively, noting that $Q = 0$.

(g) Follows from (f), making use of the obtained properties of T , S and P . \square

4. Fundamental tensors associated with the Berwald connection

In this section, we investigate the fundamental properties of the most important geometric objects associated with Berwald connection.

The following theorem guarantees the existence and uniqueness of the Berwald connection.

Theorem 4.1. [12] *Let (M, L) be a Finsler manifold. There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

- (i) $D_{h^\circ X}^\circ L = 0,$
- (ii) D° is torsion-free: $\mathbf{T}^\circ = 0,$
- (iii) The (v)hv-torsion tensor \widehat{P}° of D° vanishes: $\widehat{P}^\circ(\overline{X}, \overline{Y}) = 0.$

Such a connection is called the Berwald connection associated with the Finsler manifold (M, L) .

Moreover, the nonlinear connection associated with the Berwald connection D° coincides with the Barthel connection: $\Gamma_{D^\circ} = [J, G]$. Consequently, $\beta^\circ = \beta$ and $K^\circ = K$.

Theorem 4.2. [12] *The Berwald connection D° is explicitly expressed in terms of the Cartan connection ∇ in the form:*

$$D_X^\circ \bar{Y} = \nabla_X \bar{Y} + \hat{P}(\rho X, \bar{Y}) - T(KX, \bar{Y}). \quad (4.1)$$

In particular, we have

$$(a) \quad D_{\gamma \bar{X}}^\circ \bar{Y} = \nabla_{\gamma \bar{X}} \bar{Y} - T(\bar{X}, \bar{Y}).$$

$$(b) \quad D_{\beta \bar{X}}^\circ \bar{Y} = \nabla_{\beta \bar{X}} \bar{Y} + \hat{P}(\bar{X}, \bar{Y}).$$

Concerning the metricity of the Berwald connection, we have

Lemma 4.3. *For the Berwald connection D° , we have*

$$(a) \quad (D_{\gamma \bar{X}}^\circ g)(\bar{Y}, \bar{Z}) = 2T(\bar{X}, \bar{Y}, \bar{Z}),$$

$$(b) \quad (D_{\beta \bar{X}}^\circ g)(\bar{Y}, \bar{Z}) = -2\hat{P}(\bar{X}, \bar{Y}, \bar{Z})$$

Proposition 4.4. *The v-curvature S° of the Berwald connection vanishes: $S^\circ = 0$.*

Proof. Since D° is regular with $\mathbf{T}^\circ = 0$, the result follows from Proposition 2.5(a). \square

Theorem 4.5. *The hv-curvature tensor P° of the Berwald connection has the properties:*

$$(a) \quad \hat{P}^\circ(\bar{X}, \bar{Y}) = 0,$$

$$(b) \quad P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + P^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) = 2(D_{\beta \bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) + 2(D_{\gamma \bar{Y}}^\circ \hat{P})(\bar{X}, \bar{Z}, \bar{W}),$$

$$(c) \quad P^\circ(\bar{X}, \bar{Y})\bar{Z} = P^\circ(\bar{Z}, \bar{Y})\bar{X},$$

$$(d) \quad (D_{\gamma \bar{X}}^\circ P^\circ)(\bar{Y}, \bar{Z}, \bar{W}) = (D_{\gamma \bar{Z}}^\circ P^\circ)(\bar{Y}, \bar{X}, \bar{W}),$$

(e) P° is totally symmetric,

$$(f) \quad (D_{\gamma \bar{Y}}^\circ P^\circ)(\bar{X}, \bar{Y}, \bar{Z}) = -P^\circ(\bar{X}, \bar{Y})\bar{Y}.$$

Proof.

(b) We successively use Lemma 2.4(c) (for $X = \beta \bar{X}, Y = \gamma \bar{Y}$), Lemma 4.3, Lemma 2.3(b) and finally Theorem 4.1(iii). In fact,

$$\begin{aligned} P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + P^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) &= \beta \bar{X} \cdot (D_{\gamma \bar{Y}}^\circ g)(\bar{Z}, \bar{W}) - (D_{\gamma \bar{Y}}^\circ g)(D_{\beta \bar{X}}^\circ \bar{Z}, \bar{W}) \\ &\quad - (D_{\gamma \bar{Y}}^\circ g)(\bar{Z}, D_{\beta \bar{X}}^\circ \bar{W}) - \gamma \bar{Y} \cdot (D_{\beta \bar{X}}^\circ g)(\bar{Z}, \bar{W}) \\ &\quad + (D_{\beta \bar{X}}^\circ g)(D_{\gamma \bar{Y}}^\circ \bar{Z}, \bar{W}) + (D_{\beta \bar{X}}^\circ g)(\bar{Z}, D_{\gamma \bar{Y}}^\circ \bar{W}) \\ &\quad - (D_{[\beta \bar{X}, \gamma \bar{Y}]}^\circ g)(\bar{Z}, \bar{W}) \\ &= \beta \bar{X} \cdot (2T(\bar{Y}, \bar{Z}, \bar{W})) - 2T(\bar{Y}, D_{\beta \bar{X}}^\circ \bar{Z}, \bar{W}) \\ &\quad - 2T(\bar{Y}, \bar{Z}, D_{\beta \bar{X}}^\circ \bar{W}) - \gamma \bar{Y} \cdot (-2\hat{P}(\bar{X}, \bar{Z}, \bar{W})) \\ &\quad - 2\hat{P}(\bar{X}, D_{\gamma \bar{Y}}^\circ \bar{Z}, \bar{W}) - 2\hat{P}(\bar{X}, \bar{Z}, D_{\gamma \bar{Y}}^\circ \bar{W}) \\ &\quad - 2T(D_{\beta \bar{X}}^\circ \bar{Y}, \bar{Z}, \bar{W}) - 2\hat{P}(D_{\gamma \bar{Y}}^\circ \bar{X}, \bar{Z}, \bar{W}) \\ &= 2(D_{\beta \bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) + 2(D_{\gamma \bar{Y}}^\circ \hat{P})(\bar{X}, \bar{Z}, \bar{W}). \end{aligned}$$

(c) and (d) follow from Proposition 2.5(b) and Proposition 2.6(b) respectively, taking Proposition 4.4 and the properties of D° into account.

(e) Follows from (d) by setting $\overline{W} = \overline{\eta}$, taking into account (a) and (c).

(f) Follows from (d) by setting $\overline{X} = \overline{\eta}$ and using (a) and (e). \square

Theorem 4.6. *The h-curvature tensor R° of the Berwald connection has the properties :*

(a) $R^\circ(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R^\circ(\overline{Y}, \overline{X}, \overline{Z}, \overline{W}),$

(b) $\widehat{R}^\circ(\overline{X}, \overline{Y}) = \widehat{R}(\overline{X}, \overline{Y}) = -K\mathfrak{R}(\beta\overline{X}, \beta\overline{Y}),$

(c) $R^\circ(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + R^\circ(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) = 2\mathfrak{U}_{\overline{X}, \overline{Y}}\{(D_{\beta\overline{Y}}^\circ \widehat{P})(\overline{X}, \overline{Z}, \overline{W})\} - 2T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}),$

(d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{R^\circ(\overline{X}, \overline{Y})\overline{Z}\} = 0,$

(e) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(D_{\beta\overline{X}}^\circ R^\circ)(\overline{Y}, \overline{Z}, \overline{W}) + P^\circ(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W}\} = 0,$

(f) $(D_{\gamma\overline{X}}^\circ R^\circ)(\overline{Y}, \overline{Z}, \overline{W}) = (D_{\beta\overline{Z}}^\circ P^\circ)(\overline{Y}, \overline{X}, \overline{W}) - (D_{\beta\overline{Y}}^\circ P^\circ)(\overline{Z}, \overline{X}, \overline{W}),$

(g) $(D_{\gamma\overline{\eta}}^\circ R^\circ)(\overline{X}, \overline{Y}, \overline{Z}) = 0,$

(h) $\widehat{R}^\circ(\overline{X}, \overline{Y}) = \frac{1}{3}\{(D_{\gamma\overline{X}}^\circ H)(\overline{Y}) - (D_{\gamma\overline{Y}}^\circ H)(\overline{X})\}; H(\overline{X}) := \widehat{R}^\circ(\overline{\eta}, \overline{X}),$

(i) $R^\circ(\overline{X}, \overline{Y})\overline{Z} = (D_{\gamma\overline{Z}}^\circ \widehat{R}^\circ)(\overline{X}, \overline{Y}).$

Proof.

(b) Follows from Theorem 3.6(c) together with the identity $\widehat{R}^\circ = \widehat{R}$ [13].

(c) We use successively Lemma 2.4(c) (for $X = \beta\overline{X}, Y = \beta\overline{Y}$), the property (b) above, Lemma 4.3, Lemma 2.3(a) and finally Theorem 4.1(ii). In fact,

$$\begin{aligned} R^\circ(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + R^\circ(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) &= \mathfrak{U}_{\overline{X}, \overline{Y}}\{\beta\overline{X} \cdot (D_{\beta\overline{Y}}^\circ g)(\overline{Z}, \overline{W}) - (D_{\beta\overline{Y}}^\circ g)(D_{\beta\overline{X}}^\circ \overline{Z}, \overline{W}) \\ &\quad - (D_{\beta\overline{Y}}^\circ g)(\overline{Z}, D_{\beta\overline{X}}^\circ \overline{W})\} - (D_{[\beta\overline{X}, \beta\overline{Y}]}^\circ g)(\overline{Z}, \overline{W}) \\ &= \mathfrak{U}_{\overline{X}, \overline{Y}}\{\beta\overline{X} \cdot (-2\widehat{P}(\overline{Y}, \overline{Z}, \overline{W})) + 2\widehat{P}(\overline{Y}, D_{\beta\overline{X}}^\circ \overline{Z}, \overline{W}) \\ &\quad + 2\widehat{P}(\overline{Y}, \overline{Z}, D_{\beta\overline{X}}^\circ \overline{W})\} - 2T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}) \\ &\quad + 2\widehat{P}(D_{\beta\overline{X}}^\circ \overline{Y} - D_{\beta\overline{Y}}^\circ \overline{X}, \overline{Z}, \overline{W}) \\ &= 2\mathfrak{U}_{\overline{X}, \overline{Y}}\{(D_{\beta\overline{Y}}^\circ \widehat{P})(\overline{X}, \overline{Z}, \overline{W})\} - 2T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}). \end{aligned}$$

(d) Follows from Proposition 2.5(c), taking into account the fact that $\mathbf{T}^\circ = 0$.

(e) Follows from Proposition 2.6(d) together with the fact that $Q^\circ = 0$ and $\widehat{R}^\circ = \widehat{R}$.

(f) Follows from Proposition 2.6(c), noting that $\mathbf{T}^\circ = \widehat{P}^\circ = S^\circ = 0$.

(g) Follows from (f) by setting $\overline{X} = \overline{\eta}$, making use of Theorem 4.5(a), (e) and the fact that $K \circ \beta = 0$.

(h) We have, by (f) and Theorem 4.5,

$$(D_{\gamma\overline{X}}^\circ R^\circ)(\overline{Y}, \overline{Z}, \overline{\eta}) = 0. \quad (4.2)$$

Setting $\bar{Y} = \bar{\eta}$ in (4.2), noting that $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, we get

$$\begin{aligned}
0 &= (D_{\gamma\bar{X}}^\circ R^\circ)(\bar{\eta}, \bar{Z}, \bar{\eta}) \\
&= D_{\gamma\bar{X}}^\circ R^\circ(\bar{\eta}, \bar{Z})\bar{\eta} - R^\circ(D_{\gamma\bar{X}}^\circ \bar{\eta}, \bar{Z})\bar{\eta} - R^\circ(\bar{\eta}, D_{\gamma\bar{X}}^\circ \bar{Z})\bar{\eta} - R^\circ(\bar{\eta}, \bar{Z})D_{\gamma\bar{X}}^\circ \bar{\eta} \\
&= D_{\gamma\bar{X}}^\circ H(\bar{Z}) - \widehat{R}^\circ(\bar{X}, \bar{Z}) - H(D_{\gamma\bar{X}}^\circ \bar{Z}) - R^\circ(\bar{\eta}, \bar{Z})\bar{X} \\
&= (D_{\gamma\bar{X}}^\circ H)(\bar{Z}) - \widehat{R}^\circ(\bar{X}, \bar{Z}) - R^\circ(\bar{\eta}, \bar{Z})\bar{X}.
\end{aligned}$$

From which, making use of (d), we obtain

$$\begin{aligned}
(D_{\gamma\bar{X}}^\circ H)(\bar{Y}) - (D_{\gamma\bar{Y}}^\circ H)(\bar{X}) &= \widehat{R}^\circ(\bar{X}, \bar{Y}) + R^\circ(\bar{\eta}, \bar{Y})\bar{X} - \widehat{R}^\circ(\bar{Y}, \bar{X}) - R^\circ(\bar{\eta}, \bar{X})\bar{Y} \\
&= 2\widehat{R}^\circ(\bar{X}, \bar{Y}) - R^\circ(\bar{Y}, \bar{\eta})\bar{X} - R^\circ(\bar{\eta}, \bar{X})\bar{Y} = 3\widehat{R}^\circ(\bar{X}, \bar{Y}).
\end{aligned}$$

(i) From (4.2), noting that $K \circ \gamma = id_{\mathfrak{X}(\pi(M))}$, we get

$$\begin{aligned}
0 &= (D_{\gamma\bar{X}}^\circ R^\circ)(\bar{Y}, \bar{Z}, \bar{\eta}) \\
&= D_{\gamma\bar{X}}^\circ R^\circ(\bar{Y}, \bar{Z})\bar{\eta} - R^\circ(D_{\gamma\bar{X}}^\circ \bar{Y}, \bar{Z})\bar{\eta} - R^\circ(\bar{Y}, D_{\gamma\bar{X}}^\circ \bar{Z})\bar{\eta} - R^\circ(\bar{Y}, \bar{Z})D_{\gamma\bar{X}}^\circ \bar{\eta} \\
&= (D_{\gamma\bar{X}}^\circ \widehat{R}^\circ)(\bar{Y}, \bar{Z}) - R^\circ(\bar{Y}, \bar{Z})\bar{X}.
\end{aligned}$$

This completes the proof. \square

We terminate this section by the following

Theorem 4.7. *The following assertions are equivalent:*

- (a) *The curvature tensor \mathfrak{R} of Barthel connection vanishes.*
- (b) *The h-curvature tensor R° of Berwald connection vanishes.*
- (c) *The (v)h-torsion tensor \widehat{R}° of Berwald connection vanishes.*
- (d) *The (v)h-torsion tensor \widehat{R} of Cartan connection vanishes.*
- (e) *The π -tensor field H vanishes.*
- (f) *The horizontal distribution is completely integrable.*

Proof. These equivalences are realized by the properties (b), (h) and (i) of Theorem 4.6, taking into account that $\mathfrak{R}(X, Y) = -v[hX, hY]$ [11]. \square

5. Fundamental tensors associated with the Chern connection

In this section, we introduce and investigate the fundamental properties of the most important tensors associated with the Chern connection.

The following theorem guarantees the existence and uniqueness of the Chern connection.

Theorem 5.1. [13] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . There exists a unique regular connection D° on $\pi^{-1}(TM)$ such that*

(i) $(D_X^\circ g)(\rho Y, \rho Z) = 2g(T(K^\circ X, \rho Y), \rho Z),$

(ii) D° is torsion free: $\mathbf{T}^\circ = 0,$

where T is the (h)hv-torsion of the Cartan connection and K° is the connection map of D° .

This connection is called the Chern (Rund) connection associated with (M, L) .

Moreover, the nonlinear connection associated with the Chern connection D° coincides with the Barthel connection: $\Gamma_{D^\circ} = [J, G]$. Consequently, $\beta^\circ = \beta$ and $k^\circ = k$.

Theorem 5.2. [13] *The Chern connection D° is given in terms of the Cartan connection ∇ (or the Berwald connection D°) by:*

$$D_X^\circ \bar{Y} = \nabla_X \bar{Y} - T(KX, \bar{Y}) = D_X^\circ \bar{Y} - \hat{P}(\rho X, \bar{Y}).$$

In particular, we have

(a) $D_{\gamma \bar{X}}^\circ \bar{Y} = \nabla_{\gamma \bar{X}} \bar{Y} - T(\bar{X}, \bar{Y}) = D_{\gamma \bar{X}}^\circ \bar{Y}.$

(b) $D_{\beta \bar{X}}^\circ \bar{Y} = \nabla_{\beta \bar{X}} \bar{Y} = D_{\beta \bar{X}}^\circ \bar{Y} - \hat{P}(\bar{X}, \bar{Y}).$

Concerning the metricity of Chern connection, we have

Lemma 5.3. *For the Chern connection D° , we have*

(a) $(D_{\gamma \bar{X}}^\circ g)(\bar{Y}, \bar{Z}) = 2T(\bar{X}, \bar{Y}, \bar{Z}),$

(b) $D_{\beta \bar{X}}^\circ g = 0.$

Lemma 5.4. *The v-curvature S° of the Chern connection vanishes: $S^\circ = 0$.*

Proof. The proof is similar to that of proposition 4.4. □

Theorem 5.5. *The hv-curvature P° of the Chern connection has the properties:*

(a) $P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + P^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) = 2(D_{\beta \bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) - 2T(\hat{P}^\circ(\bar{X}, \bar{Y}), \bar{Z}, \bar{W}),$

(b) $P^\circ(\bar{X}, \bar{Y})\bar{Z} = P^\circ(\bar{Z}, \bar{Y})\bar{X},$

(c) $P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = (D_{\beta \bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) + (D_{\beta \bar{Z}}^\circ T)(\bar{Y}, \bar{W}, \bar{X}) - (D_{\beta \bar{W}}^\circ T)(\bar{Y}, \bar{X}, \bar{Z})$
 $+ T(\hat{P}^\circ(\bar{W}, \bar{Y}), \bar{X}, \bar{Z}) - T(\hat{P}^\circ(\bar{X}, \bar{Y}), \bar{Z}, \bar{W}) - T(\hat{P}^\circ(\bar{Z}, \bar{Y}), \bar{W}, \bar{X}),$

(d) $\hat{P}^\circ(\bar{\eta}, \bar{X}) = 0,$

(e) $\hat{P}^\circ(\bar{X}, \bar{Y}) = \hat{P}(\bar{X}, \bar{Y}) = (D_{\beta \bar{\eta}}^\circ T)(\bar{X}, \bar{Y}),$

(f) \hat{P}° is symmetric,

(g) $P^\circ(\bar{X}, \bar{\eta})\bar{Y} = 0, \quad P^\circ(\bar{\eta}, \bar{X})\bar{Y} = (D_{\beta \bar{\eta}}^\circ T)(\bar{X}, \bar{Y}),$

(h) $(D_{\gamma \bar{X}}^\circ P^\circ)(\bar{Z}, \bar{Y}, \bar{W}) = (D_{\gamma \bar{Y}}^\circ P^\circ)(\bar{Z}, \bar{X}, \bar{W}),$

$$(i) (D_{\gamma\bar{\eta}}^\circ P^\circ)(\bar{X}, \bar{Y}, \bar{Z}) = -P^\circ(\bar{X}, \bar{Y})\bar{Z}.$$

Proof.

(a) By Lemma 2.4(c), together with Theorem 5.1(i), we get

$$\begin{aligned} g(\mathbf{K}^\circ(X, Y)\bar{Z}, \bar{W}) + g(\mathbf{K}^\circ(X, Y)\bar{W}, \bar{Z}) &= \\ &= 2\mathfrak{U}_{X, Y}\{X \cdot g(T(KY, \bar{W}), \bar{Z}) + g(T(KX, D_Y^\circ \bar{W}), \bar{Z}) \\ &\quad + g(T(KX, \bar{W}), D_Y^\circ \bar{Z})\} - 2g(T(K[X, Y], \bar{W}), \bar{Z}). \end{aligned} \quad (5.1)$$

From which, by setting $X = \beta\bar{X}$ and $Y = \gamma\bar{Y}$ in (5.1) and using Lemma 2.3, noting that $\mathbf{T}^\circ = 0$, we get

$$\begin{aligned} P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) + P^\circ(\bar{X}, \bar{Y}, \bar{W}, \bar{Z}) &= 2\beta\bar{X} \cdot T(\bar{Y}, \bar{W}, \bar{Z}) - 2T(\bar{Y}, D_{\beta\bar{X}}^\circ \bar{W}, \bar{Z}) \\ &\quad - 2T(\bar{Y}, \bar{W}, D_{\beta\bar{X}}^\circ \bar{Z}) - 2T(\widehat{P}^\circ(\bar{X}, \bar{Y}), \bar{W}, \bar{Z}) \\ &\quad - 2T(D_{\beta\bar{X}}^\circ \bar{Y}, \bar{W}, \bar{Z}). \end{aligned}$$

Hence, the result follows.

(b) Follows from Proposition 2.5(b), making use of the hypothesis that $\mathbf{T}^\circ = 0$.

(c) Firstly, one can easily show that

$$(D_{\beta\bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) = g((D_{\beta\bar{X}}^\circ T)(\bar{Y}, \bar{Z}), \bar{W}). \quad (5.2)$$

Cyclic permutation on $\bar{X}, \bar{Z}, \bar{W}$ in the formula given by (a) above yields three equations. Adding two of these equations and subtracting the third, taking into account (5.2) and the property (b), gives

$$\begin{aligned} P^\circ(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= (D_{\beta\bar{X}}^\circ T)(\bar{Y}, \bar{Z}, \bar{W}) + (D_{\beta\bar{Z}}^\circ T)(\bar{Y}, \bar{W}, \bar{X}) - (D_{\beta\bar{W}}^\circ T)(\bar{Y}, \bar{X}, \bar{Z}) \\ &\quad + T(\widehat{P}^\circ(\bar{W}, \bar{Y}), \bar{X}, \bar{Z}) - T(\widehat{P}^\circ(\bar{X}, \bar{Y}), \bar{Z}, \bar{W}) - T(\widehat{P}^\circ(\bar{Z}, \bar{Y}), \bar{W}, \bar{X}). \end{aligned} \quad (5.3)$$

(d) Follows from (c) by setting $\bar{X} = \bar{\eta}$ and $\bar{Z} = \bar{\eta}$ and making use of the properties of the (h)hv-torsion T .

(e) Follows from (c) by setting $\bar{Z} = \bar{\eta}$, taking (d), the properties of T and the identity $D_{\beta\bar{X}}^\circ \bar{Y} = \nabla_{\beta\bar{X}} \bar{Y}$ into account.

(f) Follows from (e) and the symmetry of T .

(g) Follows from (c) by setting $\bar{Y} = \bar{\eta}$ (resp. $\bar{X} = \bar{\eta}$), making use of the obtained properties of \widehat{P}° and T .

(h) Follows from Proposition 2.6(b), taking into account that $S^\circ = T^\circ = 0$.

(i) Follows from (h) by setting $\bar{X} = \bar{\eta}$ and making use of (g). \square

Corollary 5.6. *Let (M, L) be a Finsler manifold. The following assertions are equivalent.*

(a) *The hv-curvature tensor P vanishes: $P = 0$,*

(b) *The (v)hv-torsion tensor \widehat{P} vanishes: $\widehat{P} = 0$.*

(c) *The (v)hv-torsion tensor \widehat{P}° vanishes: $\widehat{P}^\circ = 0$.*

Proof.

(a) \implies (b): Trivial.

(b) \implies (a): Suppose that \widehat{P} vanishes. From Theorem 3.4(i), we have

$$\begin{aligned} (\nabla_{\beta\overline{Z}}S)(\overline{X}, \overline{Y}, \overline{W}) &= (\nabla_{\gamma\overline{X}}P)(\overline{Z}, \overline{Y}, \overline{W}) - (\nabla_{\gamma\overline{Y}}P)(\overline{Z}, \overline{X}, \overline{W}) - S(\widehat{P}(\overline{Z}, \overline{Y}), \overline{X})\overline{W} \\ &\quad + S(\widehat{P}(\overline{Z}, \overline{X}), \overline{Y})\overline{W} - P(T(\overline{Y}, \overline{Z}), \overline{X})\overline{W} + P(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W}. \end{aligned}$$

Setting $\overline{W} = \overline{\eta}$ in the above relation, taking into account that $\widehat{S} = 0$, we get

$$P(\overline{X}, \overline{Y})\overline{Z} = P(\overline{X}, \overline{Z})\overline{Y}. \quad (5.4)$$

On the other hand, from Theorem 3.5(c), making use of the given assumption, we have

$$P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = g((\nabla_{\beta\overline{Z}}T)(\overline{X}, \overline{Y}), \overline{W}) - g((\nabla_{\beta\overline{W}}T)(\overline{X}, \overline{Y}), \overline{Z}). \quad (5.5)$$

From which, together with (5.4) and $g((\nabla_{\beta\overline{W}}T)(\overline{X}, \overline{Y}), \overline{Z}) = g((\nabla_{\beta\overline{W}}T)(\overline{X}, \overline{Z}), \overline{Y})$, we obtain

$$(\nabla_{\beta\overline{Z}}T)(\overline{X}, \overline{Y}) = (\nabla_{\beta\overline{Y}}T)(\overline{X}, \overline{Z})$$

Now, again from (5.5) the result follows.

(b) \iff (c): Follows from Theorem 5.5(e). \square

Theorem 5.7. *The h-curvature tensor R^\diamond of the Chern connection has the properties :*

- (a) $R^\diamond(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R^\diamond(\overline{Y}, \overline{X}, \overline{Z}, \overline{W})$,
- (b) $\widehat{R}^\diamond(\overline{X}, \overline{Y}) = \widehat{R}(\overline{X}, \overline{Y}) = -K\mathfrak{R}(\beta\overline{X}, \beta\overline{Y})$,
- (c) $R^\diamond(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R^\diamond(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) - 2T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}, \overline{W})$,
- (d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{R^\diamond(\overline{X}, \overline{Y})\overline{Z}\} = 0$,
- (e) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(D_{\beta\overline{X}}^\diamond R^\diamond)(\overline{Y}, \overline{Z}, \overline{W}) + P^\diamond(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W}\} = 0$,
- (f) $(D_{\gamma\overline{X}}^\diamond R^\diamond)(\overline{Y}, \overline{Z}, \overline{W}) + (D_{\beta\overline{Y}}^\diamond P^\diamond)(\overline{Z}, \overline{X}, \overline{W}) - (D_{\beta\overline{Z}}^\diamond P^\diamond)(\overline{Y}, \overline{X}, \overline{W})$
 $- P^\diamond(\overline{Z}, \widehat{P}(\overline{Y}, \overline{X}))\overline{W} + P^\diamond(\overline{Y}, \widehat{P}(\overline{Z}, \overline{X}))\overline{W} = 0$,
- (g) $(D_{\gamma\overline{\eta}}^\diamond R^\diamond)(\overline{X}, \overline{Y}, \overline{Z}) = 0$.

Proof.

(b) Follows from the identity $\widehat{R}^\diamond = \widehat{R}$ [13] together with Theorem 3.6(c).

(c) Follows from (5.1) by setting $X = \beta\overline{X}$ and $Y = \beta\overline{Y}$, making use of Lemma 2.3(a) and the identity $\widehat{R}^\diamond = \widehat{R}$.

(d) Follows from Proposition 2.5(c), taking into account the fact that $\mathbf{T}^\diamond = 0$.

(e) Follows from Proposition 2.6(d) together with $Q^\diamond = 0$, making use of (c) above.

(f) Follows from Proposition 2.6(c), noting that $\mathbf{T}^\diamond = S^\diamond = 0$ and $\widehat{P}^\diamond = \widehat{P}$.

(g) Follows from (f) by setting $\overline{X} = \overline{\eta}$, using the obtained properties of the hv-curvature tensor P^\diamond . \square

6. Fundamental tensors associated with the Hashiguchi connection

As in the previous section, we investigate the fundamental relations and properties of the most important tensors associated with the Hashiguchi connection.

Theorem 6.1. [13] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L . There exists a unique regular connection D^* on $\pi^{-1}(TM)$ such that*

- (i) D^* is vertically metric: $D_{\gamma\bar{X}}^* g = 0$,
- (ii) The (h)hv-torsion T^* of D^* satisfies: $g(T^*(\bar{X}, \bar{Y}), \bar{Z}) = g(T^*(\bar{X}, \bar{Z}), \bar{Y})$,
- (iii) The (h)h-torsion of D^* vanishes: $Q^* = 0$,
- (iv) The (v)hv-torsion of D^* vanishes: $\widehat{P}^* = 0$,
- (v) $D_{h^*X}^* L = 0$.

Such a connection is called the Hashiguchi connection associated with the Finsler manifold (M, L) .

Theorem 6.2. [13] *The nonlinear connection associated with the Hashiguchi connection D^* coincides with the Barthel connection: $\Gamma_{D^*} = [J, G]$. Consequently, $\beta^* = \beta$ and $K^* = K$.*

Moreover, the (h)hv-torsion of the Hashiguchi connection coincides with the (h)hv-torsion of the Cartan connection: $T^* = T$.

Theorem 6.3. [13] *The Hashiguchi connection D^* is given in terms of the Cartan connection (or the Berwald connection) by:*

$$D_X^* \bar{Y} = \nabla_X \bar{Y} + \widehat{P}(\rho X, \bar{Y}) = D_X^\circ \bar{Y} + T(KX, \bar{Y}). \quad (6.1)$$

In particular, we have

- (a) $D_{\gamma\bar{X}}^* \bar{Y} = \nabla_{\gamma\bar{X}} \bar{Y} = D_{\gamma\bar{X}}^\circ \bar{Y} + T(\bar{X}, \bar{Y})$.
- (b) $D_{\beta\bar{X}}^* \bar{Y} = \nabla_{\beta\bar{X}} \bar{Y} + \widehat{P}(\bar{X}, \bar{Y}) = D_{\beta\bar{X}}^\circ \bar{Y}$.

Concerning the metricity of Hashiguchi connection, we have

Lemma 6.4. *For the Hashiguchi connection D^* , we have*

- (a) $D_{\gamma\bar{X}}^* g = 0$,
- (b) $(D_{\beta\bar{X}}^* g)(\bar{Y}, \bar{Z}) = -2g(\widehat{P}(\bar{X}, \bar{Y}), \bar{Z})$.

Proposition 6.5. *The v-curvature S^* of the Hashiguchi connection has the properties:*

- (a) $S^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = -S^*(\bar{Y}, \bar{X}, \bar{Z}, \bar{W})$,
- (b) $S^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = -S^*(\bar{X}, \bar{Y}, \bar{W}, \bar{Z})$,
- (c) $S^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = S(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = g(T(\bar{X}, \bar{W}), T(\bar{Y}, \bar{Z})) - g(T(\bar{Y}, \bar{W}), T(\bar{X}, \bar{Z}))$,

$$(d) \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}} \{(D_{\gamma \overline{X}}^* S)(\overline{Y}, \overline{Z}, \overline{W})\} = 0,$$

$$(e) (D_{\gamma \overline{\eta}}^* S)(\overline{X}, \overline{Y}, \overline{Z}) = -2S(\overline{X}, \overline{Y})\overline{Z},$$

$$(f) (D_{\beta \overline{Z}}^* S)(\overline{X}, \overline{Y}, \overline{W}) = (D_{\gamma \overline{X}}^* P^*)(\overline{Z}, \overline{Y}, \overline{W}) - (D_{\gamma \overline{Y}}^* P^*)(\overline{Z}, \overline{X}, \overline{W}) \\ - P^*(T(\overline{Y}, \overline{Z}), \overline{X})\overline{W} + P^*(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W}.$$

Proof.

(b) Follows from Lemma 2.4(c) by setting $X = \gamma \overline{X}, Y = \gamma \overline{Y}$, taking into account the fact that $D_{\gamma \overline{X}}^* g = 0$ and making use of Lemma 2.3(c).

(c) Follows from the identity $D_{\gamma \overline{X}}^* \overline{Y} = \nabla_{\gamma \overline{X}} \overline{Y}$ (Theorem 6.3), noting that the vertical distribution is completely integrable.

(d) Since D^* is regular with $T^*(\overline{X}, \overline{\eta}) = T(\overline{X}, \overline{\eta}) = 0$, then from Proposition 2.6(a) and the fact that $S^* = S$, the result follows.

(e) Follows from (d) by setting $\overline{X} = \overline{\eta}$, taking into account the property that $S(\overline{X}, \overline{\eta})\overline{Y} = S(\overline{\eta}, \overline{X})\overline{Y} = \widehat{S}(\overline{X}, \overline{Y}) = 0$.

(f) Follows from Proposition 2.6(b), noting that $S^* = S$ and $\widehat{P}^* = 0$. \square

Theorem 6.6. *The hv-curvature tensor P^* of the Hashiguchi connection has the properties:*

$$(a) P^*(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + P^*(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) = 2(D_{\gamma \overline{Y}}^* \widehat{P})(\overline{X}, \overline{Z}, \overline{W}) + 2\widehat{P}(T(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}),$$

$$(b) \widehat{P}^* = 0,$$

$$(c) P^*(\overline{X}, \overline{Y})\overline{Z} - P^*(\overline{Z}, \overline{Y})\overline{X} = (D_{\beta \overline{Z}}^* T)(\overline{Y}, \overline{X}) - (D_{\beta \overline{X}}^* T)(\overline{Y}, \overline{Z}),$$

$$(d) P^*(\overline{X}, \overline{Y})\overline{Z} = P^*(\overline{X}, \overline{Z})\overline{Y},$$

$$(e) \widehat{P}^*(\overline{\eta}, \overline{X})\overline{Y} = -(D_{\beta \overline{\eta}}^* T)(\overline{X}, \overline{Y}), \quad \widehat{P}^*(\overline{X}, \overline{\eta})\overline{Y} = 0,$$

$$(f) (D_{\gamma \overline{\eta}}^* P^*)(\overline{X}, \overline{Y}, \overline{Z}) = -P^*(\overline{X}, \overline{Y})\overline{Z}.$$

Proof.

(a) Follows from Lemma 2.4(c) by setting $X = \beta \overline{X}, Y = \gamma \overline{Y}$, using Lemma 6.4 and Lemma 2.3(c). In fact,

$$\begin{aligned} P^*(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + P^*(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) &= -\gamma \overline{Y} \cdot (D_{\beta \overline{X}}^* g)(\overline{Z}, \overline{W}) + (D_{\beta \overline{X}}^* g)(D_{\gamma \overline{Y}}^* \overline{Z}, \overline{W}) \\ &\quad + (D_{\beta \overline{X}}^* g)(\overline{Z}, D_{\gamma \overline{Y}}^* \overline{W}) - (D_{[\beta \overline{X}, \gamma \overline{Y}]}^* g)(\overline{Z}, \overline{W}) \\ &= -\gamma \overline{Y} \cdot (-2g(\widehat{P}(\overline{X}, \overline{Z}), \overline{W})) - 2g(\widehat{P}(\overline{X}, D_{\gamma \overline{Y}}^* \overline{Z}), \overline{W}) \\ &\quad - 2g(\widehat{P}(\overline{X}, \overline{Z}), D_{\gamma \overline{Y}}^* \overline{W}) - 2g(\widehat{P}(D_{\gamma \overline{Y}}^* \overline{X}, \overline{Z}), \overline{W}) \\ &\quad + 2g(\widehat{P}(T(\overline{X}, \overline{Y}), \overline{Z}), \overline{W}) \\ &= 2(D_{\gamma \overline{Y}}^* \widehat{P})(\overline{X}, \overline{Z}, \overline{W}) + 2\widehat{P}(T(\overline{X}, \overline{Y}), \overline{Z}, \overline{W}). \end{aligned}$$

(b) Follows from Theorem 6.1(d).

(c) Follows from Proposition 2.5(b), taking into account the fact that $T^* = T$ and $\widehat{P}^* = Q^* = 0$.

(d) Follows from Proposition 6.5(f) by setting $\overline{W} = \overline{\eta}$, noting that $\widehat{S} = \widehat{P}^* = 0$ and $K \circ \beta = 0$.

(e) The first relation follows from (c) by setting $\overline{X} = \overline{\eta}$, using the obtained properties of the (h)hv-torsion T . On the other hand, the relation $P^*(\overline{X}, \overline{\eta})\overline{Y} = 0$ follows from (d) by setting $\overline{Y} = \overline{\eta}$, making use of (b).

(f) Follows from Proposition 6.5(f) by setting $\overline{X} = \overline{\eta}$, using (e) and the obtained properties of the v-curvature S . \square

Theorem 6.7. *The h-curvature tensor R^* has the properties :*

(a) $R^*(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = -R^*(\overline{Y}, \overline{X}, \overline{Z}, \overline{W}),$

(b) $R^*(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + R^*(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) = 2\mathfrak{U}_{\overline{X}, \overline{Y}}\{(D_{\beta\overline{Y}}^*\widehat{P})(\overline{X}, \overline{Z}, \overline{W})\},$

(c) $\widehat{R}^*(\overline{X}, \overline{Y}) = \widehat{R}(\overline{X}, \overline{Y}) = -K\mathfrak{A}(\beta\overline{X}, \beta\overline{Y}),$

(d) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{R^*(\overline{X}, \overline{Y})\overline{Z} - T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})\} = 0,$

(e) $\mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}}\{(D_{\beta\overline{X}}^*R^*)(\overline{Y}, \overline{Z}, \overline{W}) + P^*(\overline{X}, \widehat{R}(\overline{Y}, \overline{Z}))\overline{W}\} = 0,$

(f) $(D_{\gamma\overline{X}}^*R^*)(\overline{Y}, \overline{Z}, \overline{W}) + (D_{\beta\overline{Y}}^*P^*)(\overline{Z}, \overline{X}, \overline{W}) - (D_{\beta\overline{Z}}^*P^*)(\overline{Y}, \overline{X}, \overline{W})$
 $+ R^*(T(\overline{X}, \overline{Y}), \overline{Z})\overline{W} - S(\widehat{R}(\overline{Y}, \overline{Z}), \overline{X})\overline{W} - R^*(T(\overline{X}, \overline{Z}), \overline{Y})\overline{W} = 0,$

(g) $(D_{\gamma\overline{\eta}}^*R^*)(\overline{X}, \overline{Y}, \overline{Z}) = 0.$

Proof.

(b) Follows from Lemma 2.4(c) by setting $X = \beta\overline{X}, Y = \beta\overline{Y}$, taking Lemma 6.4 and Lemma 2.3(c) into account. In fact,

$$\begin{aligned} R^*((\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + R^*(\overline{X}, \overline{Y}, \overline{W}, \overline{Z}) &= \mathfrak{U}_{\overline{X}, \overline{Y}}\{\beta\overline{X} \cdot (D_{\beta\overline{Y}}^*g)(\overline{Z}, \overline{W}) - (D_{\beta\overline{Y}}^*g)(D_{\beta\overline{X}}^*\overline{Z}, \overline{W}) \\ &\quad - (D_{\beta\overline{Y}}^*g)(\overline{Z}, D_{\beta\overline{X}}^*\overline{W})\} - (D_{[\beta\overline{X}, \beta\overline{Y}]}^*g)(\overline{Z}, \overline{W}) \\ &= \mathfrak{U}_{\overline{X}, \overline{Y}}\{\beta\overline{X} \cdot (-2g(\widehat{P}(\overline{Y}, \overline{Z}), \overline{W})) + 2g(\widehat{P}(\overline{Y}, D_{\beta\overline{X}}^*\overline{Z}), \overline{W}) \\ &\quad + 2g(\widehat{P}(\overline{Y}, \overline{Z}), D_{\beta\overline{X}}^*\overline{W})\} + 2g(\widehat{P}(D_{\beta\overline{X}}^*\overline{Y}, \overline{Z}), \overline{W}) \\ &\quad - 2g(\widehat{P}(D_{\beta\overline{Y}}^*\overline{X}, \overline{Z}), \overline{W}) \\ &= 2\mathfrak{U}_{\overline{X}, \overline{Y}}\{(D_{\beta\overline{Y}}^*P)(\overline{X}, \overline{Z}, \overline{W})\}. \end{aligned}$$

(c) Follows from Theorem 3.6(c), taking into account that $\widehat{R}^* = \widehat{R}$ [13].

(d) Follows from Proposition 2.5(c), noting that $Q^* = 0, \widehat{R}^* = \widehat{R}$ and $T^* = T$.

(e) Follows from Proposition 2.6(d) together with $Q^* = 0$ and $\widehat{R}^* = \widehat{R}$.

(f) Follows from Proposition 2.6(c), making use of the relations $S^* = S, T^* = T, \widehat{R}^* = \widehat{R}$ and $\widehat{P}^* = Q^* = 0$.

(g) Follows from (f) by setting $\overline{X} = \overline{\eta}$ and using the obtained properties of the (h)hv-torsion T , the v-curvature S and the hv-curvature P^* . \square

Appendix. Intrinsic Comparison

The following tables establish a concise comparison concerning the canonical linear connections in Finsler geometry as well as the fundamental geometric objects associated with them.

Table 1.

connection	Cartan: ∇	Chern: D°	Hashiguchi: D^*	Berwald: D°
v-counterpart h-counterpart	$\nabla_{\gamma\bar{X}}\bar{Y}$ $\nabla_{\beta\bar{X}}\bar{Y}$	$D_{\gamma\bar{X}}^\circ\bar{Y} = \nabla_{\gamma\bar{X}}\bar{Y} - T(\bar{X}, \bar{Y})$ $D_{\beta\bar{X}}^\circ\bar{Y} = \nabla_{\beta\bar{X}}\bar{Y}$	$D_{\gamma\bar{X}}^*\bar{Y} = \nabla_{\gamma\bar{X}}\bar{Y}$ $D_{\beta\bar{X}}^*\bar{Y} = \nabla_{\beta\bar{X}}\bar{Y} + \hat{P}(\bar{X}, \bar{Y})$	$D_{\gamma\bar{X}}^\circ\bar{Y} = \nabla_{\gamma\bar{X}}\bar{Y} - T(\bar{X}, \bar{Y})$ $D_{\beta\bar{X}}^\circ\bar{Y} = \nabla_{\beta\bar{X}}\bar{Y} + \hat{P}(\bar{X}, \bar{Y})$
(h)v-torsion (h)hv-torsion (h)h-torsion	0 T 0	0 0 0	0 T 0	0 0 0
(v)v-torsion (v)hv-torsion (v)h-torsion	0 $\hat{P} = \nabla_G T$ $\hat{R} = -K\mathfrak{R}$	0 \hat{P} \hat{R}	0 0 \hat{R}	0 0 \hat{R}
v-curvature hv-curvature h-curvature	S P R	0 P° R°	S P^* R^*	0 P° R°
v-metricity h-metricity	$\nabla_{\gamma\bar{X}}g = 0$ $\nabla_{\beta\bar{X}}g = 0$	$D_{\gamma\bar{X}}^\circ g = 2g(T(\bar{X}, \cdot), \cdot)$ $D_{\beta\bar{X}}^\circ g = 0$	$D_{\gamma\bar{X}}^* g = 0$ $D_{\beta\bar{X}}^* g = -2g(\hat{P}(\bar{X}, \cdot), \cdot)$	$D_{\gamma\bar{X}}^\circ g = 2g(T(\bar{X}, \cdot), \cdot)$ $D_{\beta\bar{X}}^\circ g = -2g(\hat{P}(\bar{X}, \cdot), \cdot)$

Table 2.

connection	curvature tensors
Cartan	v-curvature: $S(\bar{X}, \bar{Y})\bar{Z} := -\nabla_{\gamma\bar{X}}\nabla_{\gamma\bar{Y}}\bar{Z} + \nabla_{\gamma\bar{Y}}\nabla_{\gamma\bar{X}}\bar{Z} + \nabla_{[\gamma\bar{X}, \gamma\bar{Y}]}\bar{Z}$. hv-curvature: $P(\bar{X}, \bar{Y})\bar{Z} := -\nabla_{\beta\bar{X}}\nabla_{\gamma\bar{Y}}\bar{Z} + \nabla_{\gamma\bar{Y}}\nabla_{\beta\bar{X}}\bar{Z} + \nabla_{[\beta\bar{X}, \gamma\bar{Y}]}\bar{Z}$. h-curvature: $R(\bar{X}, \bar{Y})\bar{Z} := -\nabla_{\beta\bar{X}}\nabla_{\beta\bar{Y}}\bar{Z} + \nabla_{\beta\bar{Y}}\nabla_{\beta\bar{X}}\bar{Z} + \nabla_{[\beta\bar{X}, \beta\bar{Y}]}\bar{Z}$.
Chern	$S^\circ(\bar{X}, \bar{Y})\bar{Z} = 0$. $P^\circ(\bar{X}, \bar{Y})\bar{Z} = P(\bar{X}, \bar{Y})\bar{Z} - T(\hat{P}(\bar{X}, \bar{Y}), \bar{Z}) + (\nabla_{\beta\bar{X}}T)(\bar{Y}, \bar{Z})$. $R^\circ(\bar{X}, \bar{Y})\bar{Z} = R(\bar{X}, \bar{Y})\bar{Z} - T(\hat{R}(\bar{X}, \bar{Y}), \bar{Z})$.
Hashiguchi	$S^*(\bar{X}, \bar{Y})\bar{Z} = S(\bar{X}, \bar{Y})\bar{Z}$. $P^*(\bar{X}, \bar{Y})\bar{Z} = P(\bar{X}, \bar{Y})\bar{Z} + \hat{P}(T(\bar{X}, \bar{Y}), \bar{Z}) + (\nabla_{\gamma\bar{Y}}\hat{P})(\bar{X}, \bar{Z})$. $R^*(\bar{X}, \bar{Y})\bar{Z} = R(\bar{X}, \bar{Y})\bar{Z} - \mathfrak{u}_{\bar{X}, \bar{Y}}\{(\nabla_{\beta\bar{X}}\hat{P})(\bar{Y}, \bar{Z}) + \hat{P}(\bar{X}, \hat{P}(\bar{Y}, \bar{Z}))\}$.
Berwald	$S^\circ(\bar{X}, \bar{Y})\bar{Z} = 0$. $P^\circ(\bar{X}, \bar{Y})\bar{Z} = P(\bar{X}, \bar{Y})\bar{Z} + (\nabla_{\gamma\bar{Y}}\hat{P})(\bar{X}, \bar{Z}) + \hat{P}(T(\bar{Y}, \bar{X}), \bar{Z}) + \hat{P}(\bar{X}, T(\bar{Y}, \bar{Z})) + (\nabla_{\beta\bar{X}}T)(\bar{Y}, \bar{Z}) - T(\bar{Y}, \hat{P}(\bar{X}, \bar{Z})) - T(\hat{P}(\bar{X}, \bar{Y}), \bar{Z})$. $R^\circ(\bar{X}, \bar{Y})\bar{Z} = R(\bar{X}, \bar{Y})\bar{Z} - T(\hat{R}(\bar{X}, \bar{Y}), \bar{Z}) - \mathfrak{u}_{\bar{X}, \bar{Y}}\{(\nabla_{\beta\bar{X}}\hat{P})(\bar{Y}, \bar{Z}) + \hat{P}(\bar{X}, \hat{P}(\bar{Y}, \bar{Z}))\}$.

References

- [1] H. Akbar-Zadeh, *Initiation to global Finsler geometry*, Elsevier, 2006.
- [2] D. Bao, S. S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, Berlin, 2000.
- [3] P. Dazord, *Propriétés globales des géodésiques des espaces de Finsler*, Thèse d'Etat, (575) Publ. Dept. Math. Lyon, 1969.
- [4] J. Grifone, *Structure presque-tangente et connexions, I*, Ann. Inst. Fourier, Grenoble, **22,1** (1972), 287-334.
- [5] ———, *Structure presque-tangente et connexions, II*, Ann. Inst. Fourier, Grenoble, **22,3** (1972), 291-338.
- [6] J. Klein and A. Voutier, *Formes extérieures génératrices de sprays*, Ann. Inst. Fourier, Grenoble, **18,1** (1968), 241-260.
- [7] M. Matsumoto, *The theory of Finsler connections*, Publication of the study group of geometry, Vol. 5, Dept. Math. Okayama Univ., 1970.
- [8] ———, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, 520 Japan, 1986.
- [9] R. Miron and M. Anastasiei, *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acad. Publ., No. 59, 1994.
- [10] A. A. Tamim, *General theory of Finsler spaces with applications to Randers spaces*, Ph. D. Thesis, Cairo University, 1991.
- [11] Nabil L. Youssef, *Sur les tenseurs de courbure de la connexion de Berwald et ses distributions de nullité*, Tensor, N. S., **36** (1982), 275-280.
- [12] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Cartan and Berwald connections in the pullback formalism*, To appear in Algebras, Groups and Geometries. ArXiv Number: 0707.1320.
- [13] ———, *A global approach to the theory of connections in the Finsler geometry*, Submitted. ArXiv Number: 0801.3220.